"Measure and Integration Theory"

Defn :- Intervals :- $[a,b]$, $[a,b)$, $(a,b]$, (a,b)

closed Interval ; semi closed semi open ; semi open semi closed ; open

$= \{x \in R : a \le x \le b\}$; $\{x \in R : a \le x < b\}$; $\{x \in R : a < x \le b\}$

i.) Interval is exist only when $a < b$:-
if $a = b$ or $a > b$ then interval does not exist

i.e., $[4,-6] \rightarrow$ does not exist
$(2,2) \rightarrow$ does not exist
$[2,4) \rightarrow$ exist

ii.) length of an interval I is denoted by $l(I)$ and is the difference of end points.

ex :- $I = [2,6]$ i.e., $l(I) = 6-2 = 4$

iii.) length of an Interval is always a non-negative real no.

iv.) Every non-empty subset of real must have a real number as its supremum and infimum.

v.) In Measure and Integration Theory, the universal set is set of real number.

i.e., $\phi^c = R - \phi = R$

i.e., $R^c = R - R = \phi$

vi.) Demorgan's law :- $(A \cup B)^c = A^c \cap B^c$

ii.) $(A \cap B)^c = A^c \cup B^c$

2nd lecture

Outer Measure :- let 'A' be a non-empty subset of Real numbers. let $\{I_n\}_{n \in N}$ be a countable collection of open intervals which covers A i.e., $A \subset \bigcup_{n \in N} I_n$.

For each such collection, consider the sum of lengths of intervals. We define outer major of A, denoted by $m^*(A)$ to be the infimum of all such sum i.e.,
$$m^*(A) = \inf_{A \subset \bigcup_{n \in N} I_n} \sum_n l(I_n)$$

Clearly, outer measure is a non-negative real number.

i.e., for any set A, $m^*(A) \geq 0$.

Note 1 :- Null set i) $m^*(\phi) = 0$ (trivial)

Note 2 :- ii) $A \subset B$ then $m^*(A) \leq m^*(B)$ ②

Proof :- $\{I_n\}_{n\in N}$ countable collection of open intervals that covers B. i.e., $B \subset \bigcup_n I_n$

$$A \subset \bigcup_n I_n \quad (\because A \subseteq B)$$

$$m^*(A) \leq \sum_n l(I_n) \quad -(i)$$

The inequality ① is true for all collection that covers B.

$$\Rightarrow \quad m^*(A) \leq \inf_{B \subset \bigcup_n I_n} \sum_n l(I_n)$$

$$\Rightarrow \quad m^*(A) \leq m^*(B)$$

H.P.

Note 3 :- Outer Measure of an Interval is its length.

i.e., $m^*(I) = l(I)$

Ex :- $I = [0,1]$

$\therefore m^*(I) = l(I) = 1-0 = 1$

Note 4 :- If A is a singleton set then outer measure of A is zero.

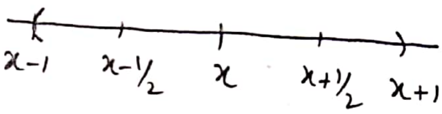
Proof :- Let $A = \{x\}$ be a singleton set.

let $I_n = (x - 1/n, x + 1/n)$ be a collection of intervals each of which covers A.

$$m^*(A) \leq \inf l(I_n)$$

$$= \lim_{n\to\infty} 2/n$$

$$= 0$$



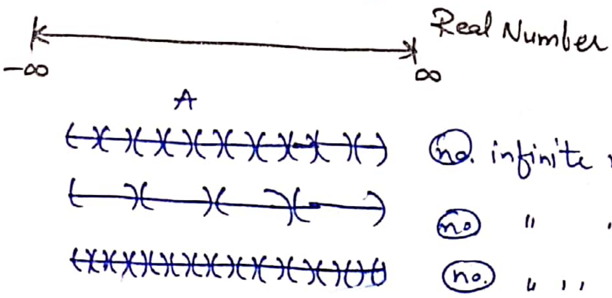
$\Rightarrow m^*(A) \leq 0$

But by defn $m^*(A) \geq 0$

So $m^*(A) = 0$

H.P.

In short 5 min lecture of Priti & Divya



no. infinite no. of intervals

no. " " " "

no. " " " "

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum_n l(I_n)$$

or

→ Outer Measure = Interval ke length ka sum jo A को cover करते।

① $m^*(\phi) = 0$ ⓘⓘ If $A \subset B$ then $m^*(A) \leq m^*(B)$

ⓘⓘⓘ If $A = \{x\}$ then $m^*(A) = 0$ ⓘⓥ $m^*(I) = l(I)$

Ex. $I = [1,2] \Rightarrow m^*(I) = 2-1 = 1$

3rd lecture

Note 5 :- If $\{A_n\}_{n\in N}$ be a countable collection of sets then

$$m^*\left(\bigcup_{n\in N} A_n\right) \le \sum_n m^*(A_n)$$

If A_n's are all disjoint then

$$m^*\left(\bigcup_{n\in N} A_n\right) = \sum_n m^*(A_n)$$

Theorem 1 :- If A is a countable set then outer measure A is equal to zero. or If A is countable then $m^*A = 0$

Proof :- We know that every countable set is the union of countable family of singletons i.e.,

$$A = \bigcup_{n\in N} \{x_n\}$$

$$\Rightarrow m^*(A) = m^*\left(\bigcup_{n\in N}\{x_n\}\right)$$

$$\Rightarrow m^*(A) \le \sum_n m^*\{x_n\}$$

$$\Rightarrow m^*(A) \le 0 \quad \text{--- (1)} \quad [\because \text{outermeasure of singleton set is zero}]$$

But

$$m^*(A) \geq 0 \quad \text{(always)}$$

So $\boxed{m^*(A) = 0}$ {Hence proved}

Note :- Each of sets N (set of Natural numbers), Z (set of Integers), Q (set of rational numbers) has outer measure zero because each of these sets is countable.

Exam 2018 2 Marks

2. The set $[0,1]$ is not countable.

Proof :- We shall prove the result by contradiction.

let, if possible, suppose that $[0,1]$ is countable.

$$\therefore m^*[0,1] = 0 \quad \text{--- (1)}$$

But

$$m^*[0,1] = l[0,1]$$
$$= 1 - 0 = 1$$

So, (1) $\Rightarrow 1 = 0$, which is not possible.

A contradiction.

Our supposition is wrong.

$\therefore [0,1]$ is not countable.

M.Imp

Theorem 2 :- Given any set A and any $\epsilon > 0$ there is an open set O s.t. $A \subset O$ and $m^*(O) < m^*(A) + \epsilon$.

Proof :- let $\{I_n\}_{n \in \mathbb{N}}$ be a countable collection of open intervals that covers A.

i.e., $A \subset \bigcup_n I_n$

$$\Rightarrow m^*(A) \leq \sum_n \ell(I_n)$$

$$\Rightarrow \sum_n \ell(I_n) < m^*(A) + \epsilon \quad \text{— (1)}$$

let

$$O = \bigcup_n I_n$$

Then O is an open set such that

$$A \subset O$$

Now,

$$m^*(O) = m^*\left(\bigcup_n I_n\right)$$
$$\leq \sum_n m^*(I_n)$$
$$= \sum_n \ell(I_n)$$
$$< m^*(A) + \epsilon \quad [\text{using (1)}]$$

Hence proved

~~[illegible]~~ point
जो अपने हर point पर
nhd हो।

$\overset{\text{nhd}}{x \in (x-\epsilon, x+\epsilon) \subseteq A}$

x — real no. ; interior point ; A — set

(a, b) ✓ $[a, b]$ ✗

$\because a \in (a-\epsilon, a+\epsilon) \not\subseteq A$

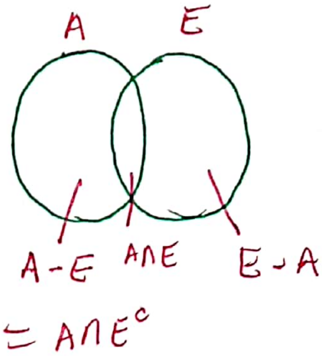
4th lecture

Defn :- Measurable Set :- A set 'E' of real numbers is said to be measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad \text{— (1)}$$

for any set A of real number.

from figure :-



$$A = (A \cap E) \cup (A \cap E^c)$$

$$\Rightarrow m^*(A) = m^*[(A \cap E) \cup (A \cap E^c)]$$
$$\leq m^*(A \cap E) + m^*(A \cap E^c)$$

The above defn. reduces to A set E is measurable.

if

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A).$$

2 Marks 2017

Theorem 3: 2018 If $m^*(E) = 0$ then E is measurable. 4 Marks

Proof: let A be any set of real number

$A \cap E \subseteq E \Rightarrow m^*(A \cap E) \leq m^*(E)$

$\Rightarrow m^*(A \cap E) \leq 0$ — (1) [$\because$ Given $m^*(E)=0$]

$A \cap E^c \subseteq A \Rightarrow m^*(A \cap E^c) \leq m^*(A)$ — (2)

Adding (1) and (2), we get

$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A)$

$\therefore$ E is measurable. H.P

Corollary 1: Every Countable set is measurable.

Proof: let E is a countable set.

from theorem (1)

$m^*(E) = 0$

from theorem (3),

E is measurable.

Note: Each of the set of Natural number, set of Integers, set of Rational number is countable and so measurable.

Corollary 2: Every singleton set is measurable.

Proof: Since outermeasure of a singleton set is zero, so every singleton set is measurable.

Theorem 4: If a set E is measurable, so is its complement.

Proof: let E be a measurable set.

To prove: E^c is measurable.

As E is a measurable set, so by definition

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

$= m^*(A \cap E^c) + m^*(A \cap E)$ [$\because$ $a+b = b+a$]

$= m^*(A \cap E^c) + m^*[A \cap (E^c)^c]$ [$\because$ $(E^c)^c = E$]

$\therefore$ E^c is measurable

H.P

Theorem 5 :- Prove that ϕ and R are measurable.

Proof :- let A be any set of real number

(a) $A \cap \phi = \phi$

$\Rightarrow m^*(A \cap \phi) = m^*(\phi)$

$\Rightarrow m^*(A \cap \phi) = 0$ — (1)

(b) $A \cap \phi^c = A \cap R$ $[\because \phi^c = R - \phi = R]$

$\Rightarrow A \cap \phi^c = A$ [Because R is universal set so $A \cap R = A$]

$\Rightarrow m^*(A \cap \phi^c) = m^*(A)$ — (2)

Adding (1) and (2),

$m^*(A \cap \phi) + m^*(A \cap \phi^c) = m^*(A)$

$\therefore \phi$ is measurable.

ii) let A be any set of real number,

(a) $A \cap R = R$

$\Rightarrow m^*(A \cap R) = m^*(R)$ — (1)

(b) $A \cap R^c = A \cap \phi$ $[\because R^c = \phi]$

$= \phi$ $[\because A \cap \phi = \phi]$

$\Rightarrow m^*(A \cap R^c) = m^*(\phi)$

$\Rightarrow m^*(A \cap R^c) = 0$ — (2)

Adding (1) and (2),

$m^*(A \cap R) + m^*(A \cap R^c) = m^*(R)$

$\therefore R$ is measurable.

4 Marks 2017

5th lecture

V. Imp ✓

Theorem 6 :- Prove that union of two measurable set is measurable. (B)

Solution :- let E_1 and E_2 be two measurable sets.

To prove :- $E_1 \cup E_2$ is measurable.

Since E_2 is measurable so by definition

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c) \quad \text{— (1)}$$

[Ist punch set of A replace from $A \cap E_1^c$]

Also we know that

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$$

[By using distributive law
II punch → add $\cap E_1^c$ in $A \cap E_2$]

Taking outermeasure, we get

$$m^*[A \cap (E_1 \cup E_2)] = m^*[(A \cap E_1) \cup (A \cap E_2 \cap E_1^c)]$$

[Change place of E_2 and E_1^c because same operation($\cap$) b/w them]

$$m^*[A \cap (E_1 \cup E_2)] = m^*[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)]$$

$$m^*[A \cap (E_1 \cup E_2)] \leq m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2)$$

Adding $m^*[A \cap (E_1 \cup E_2)^c]$ both sides, we get

$$m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c] \leq m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*[A \cap (E_1 \cup E_2)^c]$$

$$= m^*(A \cap E_1) + \boxed{m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)}$$

[By using Dmorgen]

$$= m^*(A \cap E_1) + m^*(A \cap E_1^c) \quad \text{[By using (1)]}$$

$$= m^*(A) \quad [\because \text{By definition of measurable set}]$$

So, $m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c] = m^*(A)$ H.P

Corollary 1 :- If E_1 and E_2 are measurable then $E_1 \cap E_2$ is also measurable.

Proof :- Since E_1 and E_2 are measurable.
So it follows that
E_1^c and E_2^c are also measurable.

$\Rightarrow$ $E_1^c \cup E_2^c$ is measurable.

$\Rightarrow$ $(E_1^c \cup E_2^c)^c$ is measurable.

$\Rightarrow$ $(E_1^c)^c \cap (E_2^c)^c$ is measurable.

$\Rightarrow$ $E_1 \cap E_2$ is measurable.

Corollary 2 :- If E_1 and E_2 are measurable then $E_1 - E_2$ is also measurable.

Proof :- Since E_2 is measurable so
E_2^c is also measurable

$\therefore$ $E_1 \cap E_2^c$ is also measurable.

But $E_1 - E_2 = E_1 \cap E_2^c$ $[\because A - B = A \cap B^c]$

so $E_1 - E_2$ is measurable

Theorem 7 :- Prove that symmetric difference of two measurable set is measurable.

Proof :- let E_1 and E_2 be two measurable set

To prove: $E_1 \Delta E_2$ is measurable $[\Delta = \text{Symmetric difference}]$

where $E_1 \Delta E_2 = (E_1 - E_2) \cup (E_2 - E_1)$

Now,

E_1, E_2 are measurable sets

$\Rightarrow (E_1 - E_2)$ and $(E_2 - E_1)$ are also measurable sets

$\Rightarrow (E_1 - E_2) \cup (E_2 - E_1)$ is also measurable

$\Rightarrow E_1 \Delta E_2$ is measurable

H.P

Defn :- **Translation of a set** :-

let A be any set of real numbers and x be a real number then the set
$\{a + x : a \in A\}$ is called translation of A by x.
and is denoted by $A + x$.

Note :- Translation is hold for addition & substraction but not hold for multiplication and division.

√ Imp

Theorem 8 :→ Prove that outer measure is translation (2) invarient (does not change) i.e., $m^*(A) = m^*(A+x)$

Proof : let $\{I_n\}_{n \in N}$ be a countable collection of open intervals which covers A.

i.e., $A \subset \bigcup_n I_n$ and $\sum_n l(I_n) < m^*(A) + \epsilon$ —— (1)

Also

$$(A+x) \subset \bigcup_n (I_n + x)$$

$$\Rightarrow m^*(A+x) \le m^*\left[\bigcup_n (I_n+x)\right]$$

$$\le \sum_n m^*(I_n + x)$$

$$= \sum_n l(I_n + x)$$

$$= \sum_n l(I_n)$$

[$\because$ If $I = (a,b) \Rightarrow l(I) = b-a$
$\therefore$ If $I+x = (a+x, b+x) \Rightarrow l(I+x) = b+x-a-x = b-a$]

$$\le m^*(A) + \epsilon \quad | \text{ Using (1)}$$

As ϵ is arbitrary, so

$$m^*(A+x) \le m^*(A) \quad \text{—— (2)}$$

[A+x = set translation
A = set
translation of set ≤ set]

Writing

$$A = (A+x) - x \quad \text{—— (3)}$$

Using (2), we have

$$m^*[(A+x) - x] \le m^*(A+x)$$

[Similarly, let A+x = set
(A+x)−x = translation w.r.to subtraction
so, translation ≤ set]

$$\Rightarrow m^*(A) \le m^*(A+x) \quad \text{—— (4)} \quad [\because (A+x)-x = A]$$

from (2) and (4), we get

$$\boxed{m^*(A) = m^*(A+x)}$$

Hop

{6th lecture} **{Defn}** :→ **Algebra or Boolean Algebra** :→

A collection $\mathcal{A}$ of a set X is called an Algebra of sets or Boolean Algebra if

i) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

$\mathcal{A} \to$ member is a set
$A \to$ member is element

{Note} :→ Using Demorgan's law, it follows that an Algebra is closed under finite intersection

i.e, $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

{Theorem 9} :→ The collection $\mathcal{M}$ of all measurable sets is an Algebra.

{Proof} :→ let E_1, E_2 belongs to $\mathcal{M}$ or $E_1, E_2 \in \mathcal{M}$
then E_1, E_2 are measurable sets
since union of two measurable sets is measurable.
so $E_1 \cup E_2$ is also measurable
therefore $E_1 \cup E_2 \in \mathcal{M}$.

Now let $E \in \mathcal{M}$ then E is measurable since complement of a measurable set is measurable,
so E^c is measurable.

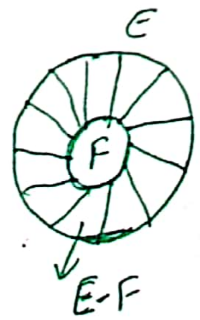
Thus $E^c \in \mathcal{M}$

Hence, $\mathcal{M}$ is an algebra.

{Theorem 10} :→ If E is a measurable set and F is a subset of E then $m^*(E-F) = m^*(E) - m^*(F)$.

{Proof} :→ Since F and $E-F$ are both disjoint sets

Sol. $E = F \cup (E-F)$ {from fig.}



$m^*(E) = m^*[F \cup (E-F)]$

$\Rightarrow m^*(E) = m^*(F) + m^*(E-F)$

$\Rightarrow m^*(E) - m^*(F) = m^*(E-F)$

$\Rightarrow m^*(E-F) = m^*(E) - m^*(F)$ {H.P}

7th lecture

Theorem :- let A be any set and $E_1, E_2, ----, E_n$ be a finite sequence of disjoint measurable sets

then

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Proof :- We shall prove the result by induction on n.

for $n = 1$, result is (trivial) → We have nothing to prove ∵ LHS = RHS.

⇒ Assume that result is true for $(n-1)$.

So,

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right] = \sum_{i=1}^{n-1} m^*(A \cap E_i) \quad \text{——— (1.)}$$

As E_n is measurable, so

$$m^*(X) = m^*(X \cap E_n) + m^*(X \cap E_n^c) \quad \text{——— (2.)}$$

where X is any set of real number.

In particular,

let, $X = A \cap \left(\bigcup_{i=1}^{n} E_i\right)$

$$\therefore \; X \cap E_n = A \cap \boxed{\left(\bigcup_{i=1}^{n} E_i\right) \cap E_n}$$

$$= A \cap E_n \quad \left[\because \bigcup_{i=1}^{n} E_i \cap E_n = E_n\right]$$

&

$$X \cap E_n^c = A \cap \boxed{\left(\bigcup_{i=1}^{n} E_i\right) \cap E_n^c}$$

$$= A \cap \left(\bigcup_{i=1}^{n-1} E_i\right) \quad \left[\because \left(\bigcup_{i=1}^{n} E_i \cap E_n^c = \bigcup_{i=1}^{n-1} E_i\right)\right]$$

So, (2) ⇒

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = m^*(A \cap E_n) + m^*\left[A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right]$$

$$= \underline{m^*(A \cap E_n)} + \sum_{i=1}^{n-1} \underline{m^*(A \cap E_i)} \quad \Big| \text{ using (1.)}$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i) \quad \left[\text{1 term \& 2 term add } 1+n-1 = n\right]$$

8th lecture

Defn : σ - Algebra or Borel Field :

Let X be a non-empty set and $\mathcal{A}$ be a non-empty collection of subsets of X.

Then $\mathcal{A}$ is called a σ-Algebra if

i) Every union of countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$.

ii) A^c is in $\mathcal{A}$ whenever A is in $\mathcal{A}$.

Note : From Demorgan's law, it follows that in a σ-Algebra, the intersection of a countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$.

8 marks 2022

✓✓ Imp

Theorem : Prove that the interval (a,∞) is measurable. (5)

Proof : Let A be any set of real numbers.

Let

$$A_1 = A \cap (a,\infty)$$

$$A_2 = A \cap (a,\infty)^c$$

→ We have to prove that $m^*(A_1) + m^*(A_2) \le m^*(A)$.

→ Let $\{I_n\}_{n\in N}$ be a countable collection of open intervals which covers A.

$$\therefore \sum_n l(I_n) < m^*(A) + \epsilon \quad \text{—— (1)}$$

Let

$$I_n' = I_n \cap (a,\infty)$$

$$I_n'' = I_n \cap (a,\infty)^c$$

Then I_n' and I_n'' are intervals (or empty) such that

$$l(I_n') + l(I_n'') = l(I_n) \quad \text{—— (2)}$$

Since, $A_1 \subset \bigcup_n I_n'$

so, $m^*(A_1) \le m^*(\bigcup_n I_n')$

$\therefore m^*(A_1) \le \sum_n m^*(I_n')$ —— (3.)

Similarly, $m^*(A_2) \le \sum_n m^*(I_n'')$ —— (4.)

Adding (3) and (4), we get

$m^*(A_1) + m^*(A_2) \le \sum_n l(I_n)$ [$\because$ using (2) and $m^*(I_n') = l(I_n')$]

$\Rightarrow m^*(A_1) + m^*(A_2) \le m^*(A) + \epsilon$ [$\because$ using (1.)]

Since ϵ is arbitrary, so

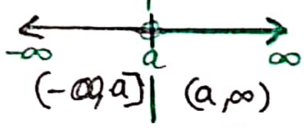
$m^*(A_1) + m^*(A_2) \le m^*(A)$.

Imp

Theorem :→ Prove that, Every open interval is measurable.

Proof :→ We know (a,∞) is measurable for each real a.

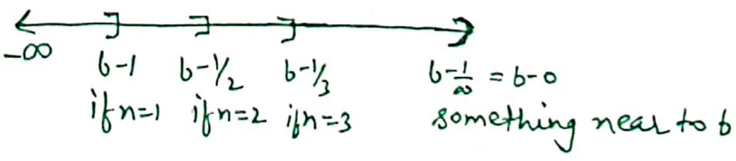
so, $(a,\infty)^c = (-\infty, a]$ is also measurable.



so, $(a,\infty)^c = (-\infty, a]$

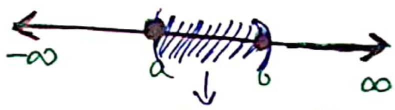
Since, $(-\infty, b] = \bigcup_{n=1}^{\infty} (-\infty, b - 1/n]$

and countable union of measurable sets is measurable, so



$(-\infty, b)$ is measurable.

$\Rightarrow (-\infty, b) \cap (a, \infty)$ is also measurable.



$\Rightarrow (a,b)$ is also measurable. Hop

Common in $(-\infty, b) \cap (a,b) = (a,b)$

If two sets is measurable then their intersection is also measurable.

9th lecture

Theorem: Prove that every open set is measurable.

Proof: We already proved every open interval is measurable. & (a,∞) is measurable, so each open set is union of a countable number of open interval, so every open set is measurable being countable union of measurable sets.

Theorem: Prove that every closed set is measurable.

Proof: We already proved (a,∞) is measurable & every open interval is measurable and also every open set is measurable, so

Every closed set is complement of open set, so it follows that every closed set is measurable because complement of a measurable set is measurable.

*** Defn: Borel Set:** A set E is called a Borel Set if it can be obtained from closed and open sets by using a finite or countable number of union and intersection operations.

M.M.M. VV Imp

Theorem: Prove that every Borel set is measurable. (6)

Proof: We firstly proved,

i) (a,∞) is measurable

ii) Every open interval is measurable

iii) Every open set is measurable

iv) Every closed set is measurable

Since a Borel set is formed by open and closed sets by taking a finite or countable number of union and intersection operation, so it follows that every Borel set is measurable.

40th lecture

Defn :- Lebesgue Measure :-

If E is a measurable set then outermeasure of E is called lebesgue Measure of E and is denoted by $m(E)$.

Ex :- We know that natural number (N) is a measurable set

$\therefore \quad m^*(N) = m(N)$.

*** Imp

Theorem :- If E_1 and E_2 are two measurable sets then prove that

(9) $$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

Proof :- As E_1 is measurable set so by definition,

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \quad \text{--- (1)}$$

where A is any set of Real number.

In particular,

$$A = E_1 \cup E_2$$

$$m^*(E_1 \cup E_2) = m^*[(E_1 \cup E_2) \cap E_1] + m^*[(E_1 \cup E_2) \cap E_1^c]$$ from (1)

$$\to m^*(E_1) + m^*[(E_1 \cup E_2) \cap E_1^c]$$

$$= m^*[(E_1 \cup E_2) \cap E_1] + m^*[(E_1 \cap E_1^c) \cup (E_2 \cap E_1^c)]$$

These steps are extra

$$= m^*[(E_1 \cup E_2) \cap E_1] + m^*[\phi \cup (E_2 \cap E_1^c)]$$

$$= m^*(E_1) + m^*[E_2 \cap E_1^c]$$

Adding $m^*(E_1 \cap E_2)$ both sides, we get

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*(E_1) + m^*(E_2 \cap E_1^c) + m^*(E_1 \cap E_2)$$

$$= m^*(E_1) + m^*(E_2 \cap E_1) + m^*(E_2 \cap E_1^c)$$ Apply interchange of second & III term & also apply commutative becz intersection always commutative

$$= m^*(E_1) + m^*(E_2)$$

$\because$ E_1 is measurable so by definition $m^*(E_2) = m^*(E_2 \cap E_1) + m^*(E_2 \cap E_1^c)$

$\therefore \; m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*(E_1) + m^*(E_2)$

As $E_1, E_2, E_1 \cap E_2, E_1 \cup E_2$ are all measurable sets so

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Imp

Theorem: If $m(E_1 \Delta E_2) = 0$ and E_1 is measurable then E_2 is also measurable moreover $mE_2 = mE_1$. (16)

Proof: Given $m(E_1 \Delta E_2) = 0$

$$\therefore \; m^*(E_1 \Delta E_2) = 0 \quad \text{—— (1)}$$

Now $E_1 - E_2 \subseteq E_1 \Delta E_2$

so,

$$m^*(E_1 - E_2) \leq m^*(E_1 \Delta E_2)$$

$$\Rightarrow m^*(E_1 - E_2) \leq 0 \quad \text{—— (2)} \quad |\text{using (1)}|$$

But outer measure is non-negative,

so

$$m^*(E_1 - E_2) \geq 0 \quad \text{—— (3)}$$

from (2) and (3),

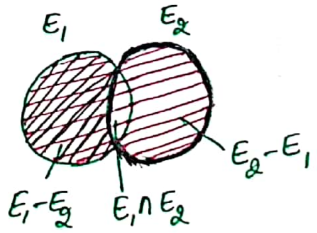
$$m^*(E_1 - E_2) = 0$$

$\Rightarrow E_1 - E_2$ is measurable.

Similarly, $E_2 - E_1$ is also measurable.

We have,

$$E_2 = [E_1 \cup (E_2 - E_1)] - (E_1 - E_2) \quad \text{—— (*)}$$



As E_1, $E_1 - E_2$, $E_2 - E_1$ are all measurable so, it follows that E_2 is measurable.

Moreover, $m^*(E_2) = m^*(E_1)$ $\left[\because m^*(E_2 - E_1) = 0 \;\&\; m^*(E_1 - E_2) = 0\right]$ so from (*)

$$m^*(E_2) = m^*(E_1) \cap m^*(E_2 - E_1) - m^*(E_1 - E_2)$$
$$= \underline{m^*(E_1)}$$

or

$$m(E_2) = m(E_1)$$

because E_2 and E_1 are both measurable.

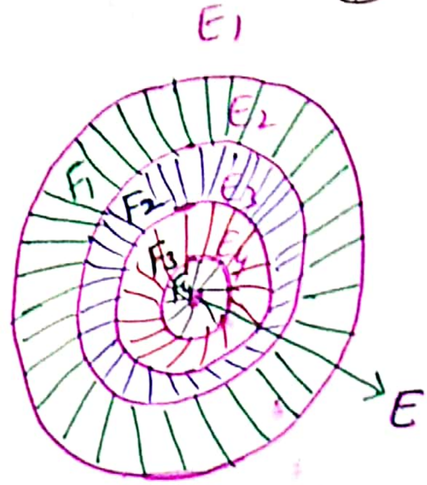
{11th lecture} VVV Imp sure. V.T. 7 marks

Theorem let $\langle E_n \rangle$ be an infinite decreasing sequence of measurable sets, that is, a sequence with $E_{n+1} \subset E_n$ for each n. (7)

let mE_1 be finite. Then

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} m E_n.$$

Proof :→ let $E = \bigcap_{i=1}^{\infty} E_i$ and



let $F_i = E_i - E_{i+1}$ —①

Then $<F_i>$ is a disjoint sequence of measurable sets.

$$\therefore m\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} m F_i \quad \text{—② }$$

But $\bigcup_{i=1}^{\infty} F_i = E_1 - E$ — ③

So,

$$m(E_1 - E) = \sum_{i=1}^{\infty} m(E_i - E_{i+1}) \quad \left[\because \text{Use eqn. ① and ③ in eqn. ②}\right]$$

$$\Rightarrow m(E_1) - m(E) = \lim_{n\to\infty} \sum_{i=1}^{n} m(E_i) - m(E_{i+1})$$

$$\Rightarrow m(E_1) - m(E) = \lim_{n\to\infty} \left[m(E_1) - \cancel{m(E_2)} + \cancel{m(E_2)} - \cancel{m(E_3)} + \cancel{m(E_3)} - m(E_4) + \cdots + m(E_n) + m(E_{n+1})\right]$$

$$\Rightarrow m(E_1) - m(E) = \lim_{n\to\infty} \left[m(E_1) - m(E_{n+1})\right]$$

$$\Rightarrow \cancel{m(E_1)} - m(E) = \cancel{m(E_1)} - \lim_{n\to\infty} m(E_{n+1}) \quad \left[\because \text{Given } mE_1 \text{ is finite}\right]$$

$$\Rightarrow m(E) = \lim_{n\to\infty} m(E_{n+1})$$

$$\Rightarrow m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m E_n \quad \left[\because n \text{ tends to } \infty \text{ so, } m(E_{n+1}) = m(E_n)\right]$$

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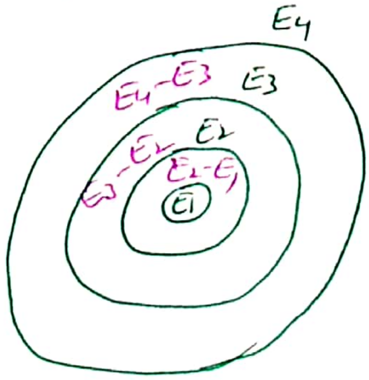
%005 ***

Theorem let $\langle E_n \rangle$ be an increasing sequence of measurable sets, that is, a sequence with $E_n \subset E_{n+1}$ for each n. (8)
let mE_1 be finite then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m E_n .$$



Proof :→ Sets $E_1, E_2-E_1, E_3-E_2, --$ are disjoint and measurable sets.

$\therefore m[E_1 \cup (E_2-E_1) \cup (E_3-E_2) \cup ---]$ [∵ Union की outermeasure sum से बराबर]

$= m(E_1) + m(E_2-E_1) + m(E_3-E_2) + --$

$\Rightarrow m[E_1 \cup (E_2-E_1) \cup (E_3-E_2) \cup --] = m(E_1) + \sum_{i=2}^{\infty} m(E_i - E_{i-1})$

But $E_1 \cup (E_2-E_1) \cup (E_3-E_2) \cup --- = \bigcup_{i=1}^{\infty} E_i$

So, $m\left(\bigcup_{i=1}^{\infty} E_i\right) = m(E_1) + \sum_{i=2}^{\infty} m(E_i - E_{i-1})$ ——— (1)

Now, $\sum_{i=2}^{\infty} m(E_i - E_{i-1}) = \lim_{n\to\infty} \sum_{i=2}^{n} m(E_i) - m(E_{i-1})$

$= \lim_{n\to\infty} \left[\cancel{m(E_2)} - m(E_1) + \cancel{m(E_3)} - \cancel{m(E_2)} + m(E_4) - \cancel{m(E_3)} + --- + m(E_n) - m(E_{n-1})\right]$

$= \lim_{n\to\infty} [m(E_n) - m(E_1)]$

$= \lim_{n\to\infty} m(E_n) - mE_1$ [∵ Given mE_1 is finite]

Putting in (1),

$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \cancel{m(E_1)} + \lim_{n\to\infty} m(E_n) - \cancel{mE_1}$

$\Rightarrow \boxed{m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m E_n}$. Proved

Defn :- **Set of the type F_σ** :-

A set E is said to be of the type F_σ if it is expressible as a union of a countable number of closed sets F_K

Exam 2018 2 Marks

i.e.,
$$\boxed{E = \bigcup_{K=1}^{\infty} F_K}$$

Defn :- **Set of the type G_δ** :-

A set E is said to be of type G_δ if it is expressible as an intersection of countable number of open sets G_K ;

i.e.,
$$\boxed{E = \bigcap_{K=1}^{\infty} G_K}$$

Clearly set of the type F_σ and G_δ are Borel sets.

→ example of Borel sets is F_σ and G_δ.

Note :-
$$(F_\sigma)^c = \left(\bigcup_{K=1}^{\infty} F_K\right)^c \quad [\because \text{By Demorgan's law}]$$
$$= \bigcap_{K=1}^{\infty} (F_K)^c$$
$$= \bigcap_{K=1}^{\infty} G_K$$
$$= G_\delta$$

[$\because F_K$ is a closed set & complement of closed set F_K is openset G_K]

Similarly, $\boxed{(G_\delta)^c = F_\sigma .}$

Theorem :- Prove that set of the type F_σ and G_δ are measurable.

Proof :- let E be a set of the type F_σ then E can be written as

$$E = \bigcup_{K=1}^{\infty} F_K \text{ where } F_K \text{ is a closed set.}$$

Since a closed set is measurable,
so each F_K is measurable.
Hence, E is measurable being countable union of measurable sets.

let E be the set of the type G_δ then
E can be written as

$$E = \bigcap_{K=1}^{\infty} G_K \text{ where } G_K \text{ is a open set.}$$

Since a open set is measurable,
so each G_K is measurable.
Hence, E is measurable being countable intersection of measurable sets.

Note :- Let A and E be subsets of real numbers and y be any real number

then

i) $A \cap (\underbrace{E+y}_{\text{translation}}) = [(A-y) \cap E] + y$ $\qquad$ $\boxed{[(A-y) \cap E] = X+y \text{ Translation}}$

ii) $A \cap (E+y)^c = [(A-y) \cap E^c] + y.$

100% V.V.Imp

{Theorem} 11 :→ If E is a measurable sets then each translate $E+y$ of E is measurable and $m(E+y) = m(E)$.

{Proof} :→ let A be any set of real numbers. Then

$$m^*[A \cap (E+y)] = m^*[\{(A-y) \cap E\} + y]$$

$$= m^*[(A-y) \cap E] \quad \text{--- (1)} \qquad \left[\begin{array}{l}\because \text{outer measure translation invarient} \\ \therefore m^*(A) = m^*(A+x)\end{array}\right]$$

and, $$m^*[A \cap (E+y)^c] = m^*[\{(A-y) \cap E^c\} + y]$$

$$= m^*[(A-y) \cap E^c] \quad \text{--- (2)} \qquad \left[\begin{array}{l}\because \text{outer measure translation invarient} \\ \therefore m^*(A+x) = m^*(A)\end{array}\right]$$

Adding (1) and (2), we get

$$m^*[A \cap (E+y)] + m^*[A \cap (E+y)^c] = m^*[(A-y) \cap E] + m^*[(A-y) \cap E^c]$$

$$\Rightarrow m^*[A \cap (E+y)] + m^*[A \cap (E+y)^c] = m^*(A-y) \qquad \left[\begin{array}{l}\because E \text{ is measurable set so by defn.} \\ m(A) = m^*(A \cap E) + m^*(A \cap E^c)\end{array}\right]$$

$$\Rightarrow m^*[A \cap (E+y)] + m^*[A \cap (E+y)^c] = m^*(A) \longrightarrow \left[\begin{array}{l}\because \text{outer measure is translation invarient} \\ m^*(A) = m^*(A-x)\end{array}\right]$$

$\therefore (E+y)$ is measurable.

Also since

$$m^*(E+y) = m^*(E) \qquad [\because \text{outermeasure is translation invarient}]$$

So, $$m(E+y) = m(E)$$

As both $E+y$ and E are measurable sets.

Defn :→ let x and y be real numbers in $[0,1)$ then sum modulo 1 of x and y is denoted by $x \dot{+} y$ and is defined by

$$x \dot{+} y = \begin{cases} x+y-1 & ; \quad x+y \geq 1 \\ x+y & ; \quad x+y < 1 \end{cases}$$

Ex :→

let $x = 0.5$
$y = 0.7$

$\therefore \quad x+y = 0.5+0.7 = 1.2 > 1$

so,

$x \dot{+} y = x+y-1 = 1.2-1 = 0.2$

Clearly, sum ~~po~~ modulo ($\dot{+}$) is commutative and associative which takes pairs of numbers in $[0,1)$ into number in $[0,1)$.

VVV Imp
500% 2017 smaller

Theorem :→ There exist a non measurable sets (12)

OR

Construct a non measurable sets.

Proof :→ for $x, y \in [0,1)$, if $x-y$ is a rational number, we say that x and y are equivalent and write $\boxed{x \sim y}$

It is clear that

Reflexive i) $x \sim x$
Symmetric ii) If $x \sim y$ then $y \sim x$
Transitive iii) $x \sim y$, $y \sim z$ then $x \sim z$

Thus '$\sim$' is an equivalence relation.
and hence partition $[0,1)$ into equivalence class such that any two elements of one class differ by a rational number while any two elements of different classes differ by an irrational number.

Let P be ~~any~~ the set which contains exactly one element from each equivalence class.
We claim that P is non measurable.
let $\{r_i\}_{i=0}^{\infty}$ be a sequence of rational numbers in $[0,1)$ with $r_0 = 0$.

Define $\quad P_i = P \dotplus r_i$

Then $\boxed{P_0 = P}$ $\quad [\because P_0 = P \dotplus r_0 \Rightarrow P_0 = P \dotplus 0 = P \; [\because r_0 = 0]]$

We claim that the sequence $\langle P_i \rangle$ be such that

i) $P_i \cap P_j = \phi$ for $i \neq j$ $\quad$ pairwise disjoint

ii) $\bigcup_i P_i = [0,1)$

For i.) On the contrary suppose that $P_i \cap P_j \neq \phi$.
let $x \in P_i \cap P_j$

Then $x \in P_i$ and $x \in P_j$

$\therefore x = p_i + r_i$ — (1) and $x = p_j + r_j$ — (2) where $p_i, p_j \in P$

from (1) and (2),

$p_i + r_i = p_j + r_j$

$\Rightarrow p_i - p_j = r_j - r_i \quad [\because r_j - r_i = \text{Rational} - \text{Rational} = \text{Rational}]$

$\Rightarrow p_i \sim p_j$

$\therefore$ p_i, p_j belongs to same class.

A contradiction, as $p_i, p_j \in P$

Hence, $P_i \cap P_j = \phi$ for $i \neq j$.

For ii) let $x \in [0,1)$ since x is in some equivalence class so it is equivalent to some element in P.

So, x differ from an element of P by a rational number r_i. Then $x \in P_i$.

Hence, $\bigcup_i P_i = [0,1)$.

Since each P_i is a translate modulo 1 of P, so each P_i will be measurable if P is measurable and

$$m(P_i) = m(P)$$

But then

$$m[0,1) = m(\bigcup_i P_i)$$

$$= \sum_{i=1}^{\infty} mP_i$$

$$= \sum_{i=1}^{\infty} mP$$

and RHS is either zero or ~~positive~~ infinite according as mP is zero or positive respectively.

This is impossible as $LHS = 1$.

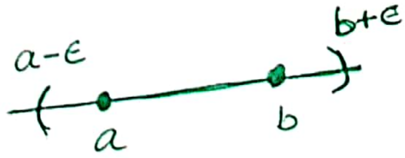
Hence, P is non measurable.

VVV Imp 500% (U.T) 10 Marks 2018 12 marks

Theorem :→ The outer measure of an interval is its length.

Proof :→ First assume that the given interval I is a bounded closed interval $[a,b]$.

Since for every positive real number ϵ, the open interval $(a-\epsilon, b+\epsilon)$ covers I, so it follows that



$$m^* I \leq l(a-\epsilon, b+\epsilon)$$

i.e., $$m^* I \leq b-a+2\epsilon$$

Since this is true for every $\epsilon > 0$, we must have

$$m^* I \leq b-a$$

i.e., $$m^* I \leq l(I) \quad \text{——} \quad (1)$$

For this special case $I = [a,b]$, it remains to prove that $m^* I \geq b-a$

let $\{I_n\}_{n \in N}$ be a countable collection of open intervals covering I, then it sufficient to establish

$$\sum_n l(I_n) \geq b-a$$

Since I is closed and bounded subset of R (set of real number), so it is compact.

By Heine Boral theorem, we can select a finite number of open intervals from $\{I_n\}$ such that their union contains I.

let these finite number of intervals be $J_1, J_2, ---, J_p$.

So it is sufficient to establish

$$\sum_{i=1}^{p} l(J_i) \geq b-a$$

Since $a \in I$, there exist an open interval (a_1, b_1) from the above mentioned finite number of intervals such that $a_1 < a < b_1$.

If $b_1 \leq b$ then $b_1 \in I$.

Since b_1 is not covered by (a_1, b_1), there is another interval (a_2, b_2) in the finite collection $J_1, J_2, --, J_p$ with $a_2 < b_1 < b_2$.

Continuing in this fashion, we obtain a sequence $(a_1, b_1), (a_2, b_2), --, (a_n, b_n)$ in the collection $J_1, J_2, --, J_p$ satisfying $a_i < b_{i-1} < b_i$ for every $i = 2, 3, --, n$.

Since the collection is finite, our process must terminate with an interval (a_n, b_n) satisfying

$$b \in (a_n, b_n)$$

Then we have,

$$\sum_n l(I_n) \geq \sum_{i=1}^{n} l(a_i, b_i)$$

$$\Rightarrow \sum_n l(I_n) = \sum_{i=1}^{n} (b_i - a_i)$$

$$\Rightarrow \sum_n l(I_n) = (b_1 - a_1) + (b_2 - a_2) + (b_3 - a_3) + \cdots + (b_{n-1} - a_{n-1}) + (b_n - a_n)$$

$$\Rightarrow \sum_n l(I_n) = b_n - (a_n - b_{n-1}) - (a_{n-1} - b_{n-2}) - \cdots (a_2 - b_1) - a_1$$

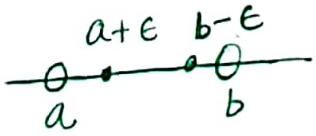
Since each expression in bracket is negative, so it follows that $\sum_n l(I_n) > b_n - a_1 > b - a$

$a_2 < b_1 < b_2$
$a_2 - b_2 = -ve$

This complete the prove when I is bounded and closed.

Next, let I be any bounded intervals with end points 'a' and 'b'.

for every real no. ϵ, we have



$$[a+\epsilon, b-\epsilon] \subset I \subset [a,b]$$

$\Rightarrow m^*[a+\epsilon, b-\epsilon] \leq m^*(I) \leq m^*[a,b]$

$\Rightarrow b-a-2\epsilon \leq m^*(I) \leq b-a$

As ϵ is arbitrary, so

$\Rightarrow b-a \leq m^*(I) \leq b-a$

$\Rightarrow \quad m^*(I) = b-a = l(I).$

Finally, suppose that I is unbounded.

then for every real number $r>0$, I contains a bounded interval H of length

$$l(H) > r$$

Therefore, $m^*(I) \geq m^*(H) \geq l(H) > r.$

Since this is true for each real number $r>0$, we must have

$$m^*(I) = \infty = l(I).$$

Hence the result.

V.V.Imp. 100%

Theorem ①:→ let $\{A_n\}$ be a countable collection of sets of real numbers then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

Proof:→ If one of the sets $\{A_n\}$ has infinite outermeasure, the inequality holds trivially.

So assume that outermeasure $\{A_n\}$ is finite for each n.

Then given $\epsilon > 0$, there exist a countable collection $\{I_{n,i}\}_i$ of open intervals such that

$$A_n \subset \bigcup_i I_{n,i}$$

$\boxed{\sum_n l(I_n) < m^*(A) + \epsilon}$

$$\text{and} \quad \sum_i l(I_{n,i}) < m^* A_n + \frac{\epsilon}{2^n} \quad -①$$

by definition of outermeasure.

Now, the collection $\{I_{n,i}\}_{n,i} = \bigcup_n \{I_{n,i}\}_i$ is countable being countable collection of countable sets and covers $\bigcup_n A_n$.

Hence,
$$
\begin{aligned}
m^*\left(\bigcup_n A_n\right) &\leq \sum_{n,i} l(I_{n,i}) \\
&= \sum_n \sum_i l(I_{n,i}) \\
&< \sum_n \left(m^* A_n + \frac{\epsilon}{2^n}\right) \quad \Big| \text{Using ①} \\
&= \sum_n m^*(A_n) + \sum_n \frac{\epsilon}{2^n} \\
&= \sum_n m^*(A_n) + \epsilon
\end{aligned}
$$

$\left(\because \sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right.$ is infinite G.P.

with $a = \frac{1}{2}$, $r = \frac{1}{2}$.

$$\left. S_\infty = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \right)$$

As ϵ is arbitrary, so

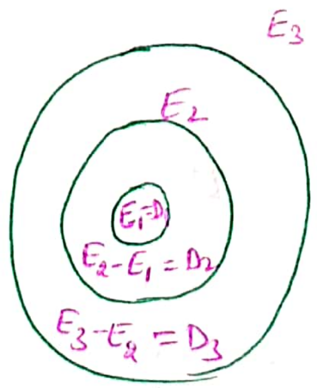
$$\boxed{m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).}$$

Imp. Result.

Theorem :→ Let $\mathcal{A}$ be an algebra of sets and $\{E_i : i \in N\}$, a sequence of set in $\mathcal{A}$ then there exist a sequence $\{D_i : i \in N\}$ of disjoint members of $\mathcal{A}$ such that $D_i \subset E_i$ and

$$\bigcup_{i \in N} D_i = \bigcup_{i \in N} E_i .$$

Proof :→



$D_1 \subset E_1$
$D_2 \subset E_2$
$D_3 \subset E_3$

$D_1, D_2, \ldots$ disjoint

$E_1 \cup E_2 \cup E_3 = E_3$
$D_1 \cup D_2 \cup D_3 = E_3$

For every $n \in N$, let

$$D_n = E_n - (E_1 \cup E_2 \cup \cdots \cup E_{n-1})$$

$$= E_n \cap E_1^c \cap E_2^c \cap \cdots \cap E_{n-1}^c \quad (2) \quad [A - B = A \cap B^c]$$

Since the complement and intersection of sets in $\mathcal{A}$ is in $\mathcal{A}$, we have $D_n \in \mathcal{A}$.

Also by construction $D_i \subset E_i$ —— (1)

let D_m and D_n be two sets.

Suppose $m<n$

Then $D_m \subset E_m$ and so [Apply $\cap D_n$ both sides]

$$D_m \cap D_n \subset E_m \cap D_n$$

$$= E_m \cap (E_n \cap E_1^c \cap \cdots \cap E_m^c \cap \cdots \cap E_{n-1}^c) \quad [\text{using } (2)]$$

$$= (E_m \cap E_m^c) \cap \cdots$$

$$= \phi \cap \cdots \qquad [A \cap A^c = \phi]$$

$$= \phi$$

$\therefore$ D_m, D_n are disjoint sets.

from (1),

$$\bigcup_i D_i \subset \bigcup_i E_i \quad \text{—— (3)}$$

It remains to prove that

$$\bigcup_i E_i \subset \bigcup_i D_i$$

for this, let $x \in \bigcup_i E_i$

Then $x \in E_n$ where 'n' is least +ve integer satisfying the condition.

$\Rightarrow x \in E_n - (E_1 \cup E_2 \cup \cdots \cup E_{n-1})$ [$x \in E_n$ इसलिये होगी मै $\notin$.]

$\Rightarrow x \in D_n$

$\Rightarrow x \in \bigcup_i D_i$

$\therefore \bigcup_i E_i \subset \bigcup_i D_i$ —— (4)

Hence, from (3) and (4), $\boxed{\bigcup_i D_i = \bigcup_i E_i}$

Imp (4)

Theorem 8 Marks 2017 :- The collection $\mathcal{M}$ of all measurable sets is a σ-algebra.

Proof :- let $E_1, E_2 \in \mathcal{M}$.

Then E_1 and E_2 are measurable sets.

Since union of two measurable set is measurable. We have, $E_1 \cup E_2$ is measurable. Thus $E_1 \cup E_2 \in \mathcal{M}$

let $E \in \mathcal{M}$. Then E is measurable.

Since complement of a measurable set is measurable
$\therefore E^c$ is measurable. So $E^c \in \mathcal{M}$.

Hence $\mathcal{M}$ is an Algebra.

To complete the proof :- We have to prove that the union of a countable collection of measurable set is measurable.

let E be the union of a countable collection of measurable sets.

Such as E can be written as a union of a sequence $\{D_n\}$ of pairwise disjoint measurable sets.

let $A \subset R$ and $E_n = \bigcup_{i=1}^{n} D_i$.

Then E_n is measurable for each n and $E^c \subset E_n^c$.

$\because E_n \subseteq E$
$\therefore E^c \subseteq E_n^c$

Therefore;

$$m^*(A) = m^*(A \cap E_n) + m^*(A \cap E_n^c)$$
$$\geq m^*(A \cap E_n) + m^*(A \cap E^c) \quad [\because E^c \subseteq E_n^c]$$

But

$$m^*(A\cap E_n) = m^*[A\cap(\bigcup_{i=1}^{n} D_i)] \qquad [\text{Given } E_n = \bigcup_{i=1}^{n} D_i]$$

$$= \sum_{i=1}^{n} m^*(A\cap D_i)$$

So, $m^*(A) \geq \sum_{i=1}^{n} m^*(A\cap D_i) + m^*(A\cap E^c)$ for all n.

Since LHS of this inequality is independent of n, we have

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A\cap D_i) + m^*(A\cap E^c)$$

By the countable subadditivity of m^*.

Now, $m^*(A) - m^*(A\cap E^c) \geq \sum_{i=1}^{\infty} m^*(A\cap D_i)$

$$\Rightarrow \quad m^*(A\cap E) = m^*[A\cap(\bigcup_{i=1}^{\infty} D_i)]$$

$$= m^*[\bigcup_{i=1}^{\infty}(A\cap D_i)]$$

$$\leq \sum_{i=1}^{\infty} m^*(A\cap D_i)$$

Here apply Countable subadditivity

$\therefore$ $m^*(A) \geq m^*(A\cap E) + m^*(A\cap E^c)$

This proves that E is measurable.

Hence $\mathcal{M}$ is a σ-algebra.

Theorem (10) let E be a set with $m^*E < \infty$. Then E is measurable iff, given $\epsilon > 0$, there is a finite union B of open intervals, such that

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$$m^*(E \Delta B) < \epsilon .$$

Proof :→ Suppose E is measurable and let $\epsilon > 0$ be given.

Then $\exists$ an open set $O \supset E$ s.t.

$$m^*(O - E) < \epsilon/2 .$$

As $m^*(E)$ is finite. so is m^*O.

Further since 'O' is open, $\exists$ a disjoint sequence $\{I_n\}$ of open intervals s.t.

$$O = \bigcup_n I_n .$$

Now, $m^* O < \infty$.

So, $m^*\left(\bigcup_n I_n\right) < \infty$

$$\Rightarrow \sum_n m^*(I_n) < \infty$$

$$\Rightarrow \sum_n l(I_n) < \infty$$

Thus $\sum_n l(I_n)$ is convergent. So $\exists$ some $k \in N$

s.t. $$\sum_{n=k+1}^{\infty} l(I_n) < \epsilon/2 .$$

let $B = \bigcup_{n=1}^{k} I_n$.

Then $B - E \subseteq O - E$ since $B \subseteq O$.

so, $m^*(B-E) \le m^*(O-E) < \epsilon/2$.

Also, $E \subseteq O$

$\Rightarrow E - B \subseteq O - B = \bigcup_{n=k+1}^{\infty} I_n$

$\Rightarrow m^*(E-B) \le m^*\left(\bigcup_{n=k+1}^{\infty} I_n\right)$

$= \sum_{n=k+1}^{\infty} m^*(I_n) = \sum_{n=k+1}^{\infty} l(I_n) < \epsilon/2$.

Hence, $m^*(E \Delta B) = m^*[(E-B) \cup (B-E)]$

$\le m^*(E-B) + m^*(B-E)$

$< \epsilon/2 + \epsilon/2$

$= \epsilon$

Conversely assume that for a given $\epsilon > 0$, there is a finite union $B = \bigcup_{n=1}^{m} I_n$ of open intervals

s.t. $m^*(E \Delta B) < \epsilon/3$.

Since $m^*E < \infty$, $\exists$ an open set O s.t. $E \subset O$ and

$m^*O < m^*E + \epsilon/3$ — (1)

let $S = \bigcup_{n=1}^{m} (I_n \cap O) = \left(\bigcup_{n=1}^{m} I_n\right) \cap O = B \cap O.$

Then $S \subseteq B$ and so

$$S \Delta E = (E - S) \cup (S - E)$$
$$\subset (E - S) \cup (B - E)$$

However,

$$E - S = E - (B \cap O)$$
$$= E \cap (B \cap O)^c$$
$$= E \cap (B^c \cup O^c)$$
$$= (E \cap B^c) \cup (E \cap O^c)$$
$$= (E \cap B^c)$$
$$= E - B \qquad [\because E \cap O^c = \phi]$$

$\therefore S \Delta E \subseteq (E - B) \cup (B - E) = E \Delta B.$

So, $m^*(S \Delta E) \leq m^*(E \Delta B) < \epsilon/3.$

Now, $E \subset S \cup (S \Delta E)$

so, $m^* E \leq m^*(S) + m^*(S \Delta E)$

$< m^*(S) + \epsilon/3$ —— (2)

Also, $O - E \subset (O - S) \cup (S \Delta E)$

So, $m^*(O - E) \leq m^*(O - S) + m^*(S \Delta E)$

$< m^*(O - S) + \epsilon/3$

Since O and S have finite measure,

$$m^*(O-E) < m^*O - m^*S + \epsilon/3$$

$$< m^*E + \epsilon/3 - m^*S + \epsilon/3 \quad [\text{Using } ①]$$

$$< m^*S + \epsilon/3 + \epsilon/3 - m^*S + \epsilon/3$$

$$= \epsilon.$$

Hence E is measurable.

Quaity Exam 2017 U.T 3 Marks

Ques: Give an example of an uncountable set with outer measure zero.

Soln: Cantor set.

The Cantor set is uncountable with outer measure zero.

The cantor set, denoted by C, is a subset of the interval [0,1].

To construct the C, we proceed as follows,

Let C_1 denote the interval [0,1].

Remove from C_1 the open interval $(\frac{1}{3}, \frac{2}{3})$, the middle third of the interval C_1 and denote the remaining set by C_2.

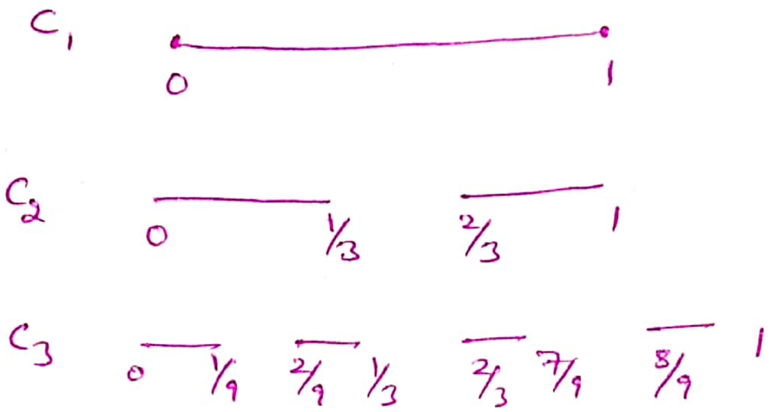
Clearly . $C_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

Next remove from C_2, the open interval $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$

the middle thirds of two closed intervals $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$ in C_2 and denote the remaining set by C_3.

We obseeve that

$$C_3 = [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1]$$



If we continue this process, at each stage, deleting the open middle third of each closed interval remained from the previous stage, we obtain a sequence $\{C_n\}$ of closed sets such that $C_{n+1} \subset C_n$.

The Cantor set C is defined as

$$C = \bigcap_{n=1}^{\infty} C_n$$

The Cantor set C is uncountable.

We show $m^* C = 0$.

let E_n denote the set composed of all the open intervals removed at the nth stage so that

$E_1 = (1/3, 2/3)$

$E_2 = (1/9, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$

$E_3 = (\frac{1}{27}, \frac{2}{27}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{19}{27}, \frac{20}{27}) \cup (\frac{25}{27}, \frac{26}{27})$

Then we observe that

$$C = [0,1] - \bigcup_{n=1}^{\infty} E_n$$

Now,

$$m^* E_1 = l(E_1) = 2/3 - 1/3 = 1/3$$

$$m^* E_2 = l(E_2) = \frac{2}{9} - \frac{1}{9} + \frac{8}{9} - \frac{7}{9} = \frac{1}{9} + 1/9 = \frac{2}{9} = \frac{2}{3^2}$$

$$m^* E_3 = l(E_3) = \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} = \frac{4}{27} = \frac{4}{3^3}$$

$$m^* E_n = l(E_n) = \frac{2^{n-1}}{3^n}$$

$$\therefore \; m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m^* E_n = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = 1$$

Thus, $m^* C = m^*[0,1] - m^*\left(\bigcup_n E_n\right) = 1 - 1 = 0$

Hence, there exist uncountable set with outermeasure zero.

4 Marks Exam 2017

Theorem :- let E be any set. Then $\exists$ a G_δ-set $G \supset E$ such that $m^* E = m^* G$.

Proof :- For each $n \in N$, $\exists$ an open set $O_n \supset E$ such that

$$m^* O_n \le m^*(E) + \frac{1}{n}$$

Define $G = \bigcap_{n=1}^{\infty} O_n$. Then G is a G_δ-set and $G \supset E$.

Moreover, $G \subseteq O_n$ for each n.

$$\therefore \; m^* G \le m^* O_n \le m^* E + \frac{1}{n} \quad \forall n$$

So, $m^* G \le m^* E$ — (1)

Also, $E \subset G$

So $m^* E \le m^* G$ — (2)

from (1) and (2), we get

$$\boxed{m^* E = m^* G}$$

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